



DERIVATION OF NONLINEAR WAVE EQUATION FOR FLEXURAL MOTIONS OF AN ELASTIC BEAM TRAVELLING IN AN AIR-FILLED TUBE

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(Received 13 December 2000; and in final form 5 November 2001)

Asymptotic analysis is carried out to derive a nonlinear wave equation for flexural motions of an elastic beam of circular cross-section travelling along the centre-axis of an air-filled, circular tube placed coaxially. Both the beam and tube are assumed to be long enough for end-effects to be ignored and the aerodynamic loading on the lateral surface of the beam is considered. Assuming a compressible inviscid fluid, the velocity potential of the air is sought systematically in the form of power series in terms of the ratios of the tube radius to a wavelength and of a typical deflection to the radius. Evaluating the pressure force acting on the lateral surface of the beam, the aerodynamic loading including the effects of finite deflection as well as of air's compressibility and axial curvature of the beam are obtained. Although the nonlinearity arises from the kinematical condition on the beam surface, it may be attributed to the presence of the tube wall. With the aerodynamic loading thus obtained, a nonlinear wave equation is derived, whereas linear theory is assumed for the flexural motions of the beam. Some discussions are given on the results.

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1. INTRODUCTION

THIS PAPER CONSIDERS NONLINEAR AERODYNAMIC LOADING on an elastic beam travelling in an air-filled tube along the axial direction to derive a wave equation for flexural motions of the beam. A study of this problem is motivated by an interest of fluid-structure interactions related to dynamics of magnetically levitated trains travelling in a long tunnel at high speed (a Mach number 0.4 or even higher). Because the trains have no mechanical supports but auxiliary guiding wheels, they are prone to destabilization by aerodynamic loading. Destabilization will give rise not only to vibrations of vehicles but also to wave propagation along the train, since trains are usually very long compared with their lateral dimension.

To make an analytical model, some simplifications are needed focusing on specific aspects of particular phenomena. One simple model is to regard the train as a spatially periodic structure consisting of many *rigid* beams of finite length articulated by elastic couplers, which provide a restoring moment proportional to difference in deflection angle at the end of two beams adjacent to each other. This *discrete* model may be appropriate for disturbances of short wavelength comparable to each vehicle's length. The other model is to regard the train as a long *elastic* beam by neglecting articulations. This *continuous* model may be good for disturbances of long wavelength. Furthermore, there are required models to treat end-effects. In the following, the continuous model is adopted by

assuming, for simplicity, that both the beam and the tube extend infinitely, to ignore the end-effects, and only effects of aerodynamic loading on the lateral surface are considered.

Here it is remarked that the continuous model may be reduced from the discrete model by taking a long-wavelength limit. In fact, if a continuum approximation is made, a flexural wave equation for a uniform beam can be derived with the bending rigidity κl , κ and l being, respectively, a spring constant of the coupler (moment/radian angle) and the length of each rigid beam. Furthermore, if each rigid beam is replaced by an *elastic* beam, i.e., the beam bending rigidity is taken into account, wave propagation exhibits a very complicated behaviour due to multiple reflection and transmission at the couplers. Wave propagation in such a spatially periodic structure may be called a Bloch wave after the name in solid state physics [see, e.g., Kittel (1976), and Sugimoto & Horioka (1995)]. In this system, there appear two modes associated with the couplers and beams. Each Bloch dispersion relation shows a banded structure in frequency with passing and stopping bands, which correspond, respectively, to propagation and attenuation. If a low-frequency and long-wavelength limit is taken, the dispersion relation for the propagation is reduced to the one of a flexural wave on a uniform beam. Thus the effect of the couplers may be neglected. Detailed discussions will be given elsewhere.

When the fluid is moving relative to the beam, there may occur not only attenuation but also growth of the waves leading to instability. Although in a different context from the present one, Howe (1986) discusses attenuation and diffraction of flexural waves at gaps in a plate placed in a still fluid. The interesting result is that when a frequency of flexural waves is higher than a coincidence frequency at which the speed of the flexural waves *in vacuo* is equal to the sound speed, no attenuation occurs. Because the speed of the flexural waves in the present problem is estimated to be slower than the sound speed, such a phenomenon is not expected to occur. Similarly, however, there will be many interesting phenomena associated with the discontinuity in the system depending on the ratio of the bending rigidity of the beam to the bending rigidity κl of the coupler, and on the relative speed. But in this paper, attention is focused on the propagating waves in the continuous model to derive a wave equation, by which their instability will be discussed.

In a train-tunnel problem, a Reynolds number is very high and a boundary-layer separation does not occur for such a slender body. Thus a compressible but inviscid flow field is assumed for the air in an annular region between the beam and the tube. When a beam executes lateral motions, the lateral force is brought about by the acceleration reaction of fluid and the beam mass is increased by the induced mass (Batchelor 1970). This force gives rise to convective instability if the beam is travelling relative to the surrounding fluid. When flexural motions of the beam are considered, effects of axial curvature and of air compressibility modify this result. For infinitesimally small deflection, it is found from the linear theory that the flexural rigidity of the beam acts to suppress the instability together with the effect of axial curvature on the induced mass (Sugimoto & Kugo 2001). However, nonlinear effects due to finite deflection are as yet unknown. To answer this question, we clarify the aerodynamic loading on an elastic beam of circular cross-section placed coaxially in a tube of circular cross-section. This leads to the derivation of a compact, nonlinear wave equation for flexural motions of an elastic beam.

In what follows, we formulate in Section 2 the problem and present the aerodynamic equations and the flexural wave equation of the beam. In Section 3, a systematic asymptotic expansion is developed in terms of two parameters measuring small but finite deflection and long but finite wavelength relative to the tube radius. No *ad hoc* assumptions are introduced except order estimation in the expansion. The aerodynamic

loading is obtained by specifying the flow field, and then a nonlinear wave equation for the flexural motions of the beam is derived. Some discussions are given in Section 4.

2. FORMULATION OF THE PROBLEM

2.1. GEOMETRICAL CONFIGURATION

Suppose that an elastic beam of radius b is placed coaxially in a rigid tube of radius $R (> b)$, both of infinite extent, and that the annular region is filled with air. Let the beam be travelling at constant, subsonic speed U in the axial direction. Take the x -axis along the centre-axis of the tube in the direction of travel, and the y - and z -axis in a plane normal to the x -axis with the origin on the centre-axis. The plane polar coordinates $(r, \theta;)$ are also used in the (y, z) plane as shown in Figure 1.

To suppress instability, we consider a restoring force on the beam, which is proportional to the magnitude of deflection, and assume that the deflection is limited to the y direction only. If a mirror image is taken with respect to the plane $z = 0$, the present configuration may model a tunnel in which a train of semi-circular cross-section travels inside a tunnel of a semi-circular one, and no vertical motions, i.e., no ground effects are considered.

2.2. AERODYNAMIC EQUATIONS

The surrounding air is assumed to be an inviscid, ideal gas. The basic equations consist of the equations of continuity, momentum and energy. No gravity is taken into account. Because viscosity is ignored, the entropy is conserved and the homentropic flow is assumed. This is substantially stipulated by use of the adiabatic relation between the

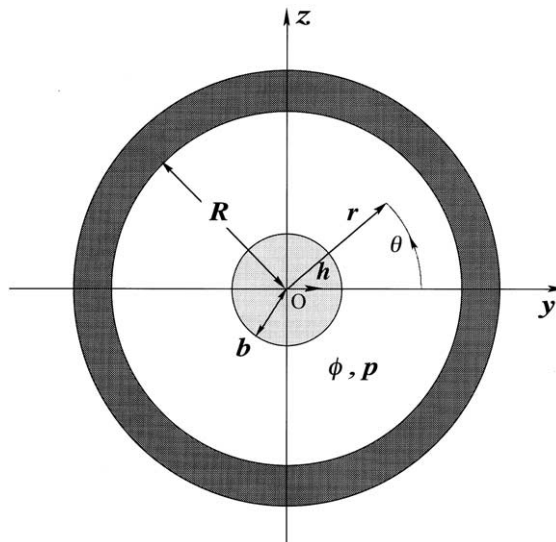


Fig. 1. Cross-sectional configuration of the tube normal to the axis of the tube where the beam of radius b is placed coaxially in the tube of radius R and the air is filled in the annular region: the x -axis is taken out of the paper while the y - and z -axis are taken in the paper and the plane polar coordinates (r, θ) are also used: the deflection of the beam, denoted by $h(x, t)$, is limited in the y direction only and the beam is subjected to a restoring force proportional to the magnitude of the deflection.

density and the pressure. Thus, the basic equations are given as follows:

$$\frac{1}{\rho} \left(\frac{\partial \rho}{\partial t} + \mathbf{u} \nabla \rho \right) + \nabla \mathbf{u} = 0, \quad (2.1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p, \quad (2.2)$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0} \right)^\gamma, \quad (2.3)$$

where ρ , \mathbf{u} and p denote, respectively, the density, the velocity vector, and the pressure in the flow field, t being the time; the subscript 0 of p and ρ designates the respective constant values in equilibrium, γ being the ratio of specific heats.

Introducing the velocity potential ϕ via $\mathbf{u} = \nabla \phi$, equations (2.1) and (2.2) are combined into a single wave equation of ϕ . Firstly, equation (2.2) is integrated by Bernoulli's theorem as

$$\frac{\partial \phi}{\partial t} + \frac{|\mathbf{u}|^2}{2} + \int \frac{dp}{\rho} = 0. \quad (2.4)$$

Next we express the derivatives of ρ divided by ρ in equation (2.1) in terms of \mathbf{u} alone. Differentiating equation (2.4) with respect to t , we have

$$\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t} \frac{|\mathbf{u}|^2}{2} + \frac{a^2}{\rho} \frac{\partial \rho}{\partial t} = 0, \quad (2.5)$$

where $a (= \sqrt{dp/d\rho} = \sqrt{\gamma p/\rho})$ is the local sound speed and is calculated by equation (2.4) with equation (2.3) in terms of ϕ as follows:

$$a^2 = a_0^2 - (\gamma - 1) \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) \quad (2.6)$$

with $a_0 (= \sqrt{\gamma p_0/\rho_0})$ being the linear sound speed. This also gives the pressure in terms of ϕ as

$$\frac{p}{p_0} = \left(\frac{a}{a_0} \right)^{2\gamma/(\gamma-1)} = \left[1 - \frac{(\gamma-1)}{a_0^2} \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) \right]^{\gamma/(\gamma-1)}. \quad (2.7)$$

To form $\mathbf{u} \nabla \rho$, we take the inner product of equation (2.2) with \mathbf{u} to obtain

$$\frac{\partial}{\partial t} \frac{|\mathbf{u}|^2}{2} + \mathbf{u} (\mathbf{u} \nabla) \mathbf{u} = -\frac{a^2}{\rho} \mathbf{u} \nabla \rho. \quad (2.8)$$

Using equations (2.5) and (2.8) to express the first term of equation (2.1) involving ρ in terms of ϕ , it follows that

$$\frac{\partial^2 \phi}{\partial t^2} - a^2 \Delta \phi = -\frac{\partial}{\partial t} |\nabla \phi|^2 - \frac{1}{2} (\nabla \phi \nabla) |\nabla \phi|^2. \quad (2.9)$$

Further, eliminating a by equation (2.6), we arrive at the wave equation for ϕ :

$$\frac{\partial^2 \phi}{\partial t^2} - a_0^2 \Delta \phi = - \frac{\partial}{\partial t} |\nabla \phi|^2 - (\gamma - 1) \frac{\partial \phi}{\partial t} \Delta \phi - \frac{1}{2} (\nabla \phi \nabla) |\nabla \phi|^2 - \frac{1}{2} (\gamma - 1) |\nabla \phi|^2 \Delta \phi. \quad (2.10)$$

2.3. FLEXURAL WAVE EQUATION

Although nonlinear aerodynamic equations are employed for the air, the elastic behaviour of the beam is assumed to be modelled by the linear theory. Introducing a new coordinate ξ moving with the beam at speed U in the positive direction of x , and identifying t with τ ,

$$\xi = x - Ut \quad \text{and} \quad \tau = t, \quad (2.11)$$

the flexural wave equation is given for deflection $H(\xi, \tau)$ as follows:

$$m \frac{\partial^2 H}{\partial \tau^2} + EI \frac{\partial^4 H}{\partial \xi^4} + KH = Q, \quad (2.12)$$

where m and EI denote, respectively, the density of the beam per unit axial length and the bending rigidity, while K and Q represent, respectively, the spring constant of the restoring force and the total pressure force acting on the lateral surface of the beam per unit axial length.

When equation (2.12) is expressed in terms of x and t , it is transformed into

$$m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 h + EI \frac{\partial^4 h}{\partial x^4} + Kh = q \quad (2.13)$$

with $h(x, t) = H(\xi, \tau)$ and $q(x, t) = Q(\xi, \tau)$. Here, q is calculated by integrating the pressure on the beam surface per unit axial length as follows:

$$q = - \int_0^{2\pi} p_b b \cos \theta' d\theta', \quad (2.14)$$

where p_b denotes the pressure on the beam surface and θ' measures a circumferential angle on the periphery of the beam's cross-section normal to the x -axis.

2.4. BOUNDARY CONDITIONS

Next we consider the boundary conditions. Because inviscid motions are assumed, the slip condition is imposed on the tube wall. It is given simply by

$$\mathbf{n} \nabla \phi = 0, \quad (2.15)$$

where \mathbf{n} is the unit vector directed inward normal to the wall. On the beam surface, the kinematical condition is imposed. Denoting the surface by $F(x, y, z, t) = 0$, it is specified formally by

$$\frac{\partial F}{\partial t} + \nabla \phi \nabla F = 0 \quad \text{on} \quad F = 0. \quad (2.16)$$

Here, F is given in terms of the unknown deflection h . We now look for its explicit dependence on h .

When the beam is bent elastically, linear theory assumes that (i) a cross-section normal to the centre-line before deflection remains normal to it after deflection, (ii) the shape of the cross-section remains unchanged and (iii) the stretching of the centre-line is negligible.

To implement these assumptions in F , let us consider the beam at a point $x = X$. When the beam is deflected, the centre-line at this point is displaced to a position at $y = Y = h(X, t)$, while the cross-section normal to the x -axis rotates about the z -axis. We take the y' -axis normal to the z -axis in the cross-section rotated with the origin on the centre-line of the beam (see Figure 2). A point on periphery of the cross-section is specified by an angle θ' measured anticlockwise from the y' -axis [Figure 2(b)]. Let the position vector directed from the centre-line at $x = X$ to the point on the periphery be denoted by \mathbf{r}' . Denoting the unit vectors in the x, y and z directions by \mathbf{i}, \mathbf{j} , and \mathbf{k} , respectively, and the unit vector in the y' direction by \mathbf{j}' , \mathbf{r}' is written as

$$\mathbf{r}' = b \cos \theta' \mathbf{j}' + b \sin \theta' \mathbf{k}. \tag{2.17}$$

The unit vector \mathbf{j}' is given by $(-\sin \alpha, \cos \alpha, 0)$ where α is the angle between the tangent to the centre-line and the x -axis so that $\tan \alpha = \partial h / \partial x$. The coordinates (x, y, z) of a point on the periphery are given by

$$\begin{pmatrix} x - X \\ y - Y \\ z \end{pmatrix} = \begin{pmatrix} \mathbf{r}' \mathbf{i} \\ \mathbf{r}' \mathbf{j} \\ \mathbf{r}' \mathbf{k} \end{pmatrix} = \begin{pmatrix} -b \cos \theta' \sin \alpha \\ b \cos \theta' \cos \alpha \\ b \sin \theta' \end{pmatrix}. \tag{2.18}$$

Since $Y = h(X, t)$, the following relation must be satisfied:

$$y - b \cos \theta' \cos \alpha = h(x + b \cos \theta' \sin \alpha), \tag{2.19}$$

with t suppressed. The function F is expressed by eliminating the parameter θ' in equation (2.19) with $z = b \sin \theta'$.

The linear theory of the beam assumes that a typical axial wavelength of deflection, denoted by λ , is much longer than a typical deflection c . Here, c is smaller than the tube radius, of course. This implies that the angle α is small and that

$$|\alpha| \approx \left| \frac{\partial h}{\partial x} \right| \approx \frac{c}{\lambda} \ll 1. \tag{2.20}$$

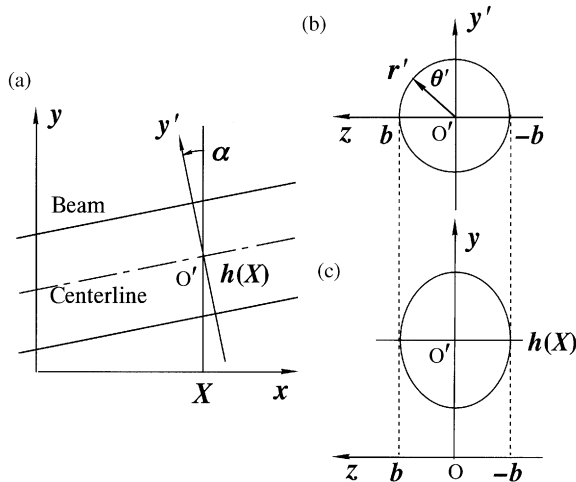


Fig. 2. Geometrical configuration of the deflected beam: (a) displays the cross-section of the tube cut by the plane $z = 0$ where a point on the centre-line of the beam at $x = X$ is deflected to a new position $y = h(X)$ and the original cross-section at $x = X$ rotates about the z -axis by angle α ; (b) displays the cross-section rotated where the y' axis is taken normal to the z -axis and θ' is the angle measured anticlockwise from the y' -axis; (c) displays the cross-section normal to the x -axis.

Furthermore from the estimation that

$$\left| b^{n-1} \frac{\partial^n h}{\partial x^n} \right| \approx \left(\frac{b}{\lambda} \right)^{n-1} \frac{c}{\lambda} \ll 1, \tag{2.21}$$

we may expand h in equation (2.19) around x , and neglect the errors of $\mathcal{O}(x^3 b/\lambda)$ as

$$y - b \cos \theta' \cos \alpha = h + \frac{\partial h}{\partial x} b \cos \theta' \sin \alpha + \mathcal{O}(x^3 b/\lambda). \tag{2.22}$$

Eliminating the parameter θ' , F is expressed in terms of h as

$$F(x, y, z, t) = \frac{(y - h)^2}{1 + (\partial h/\partial x)^2} + z^2 - b^2 = 0. \tag{2.23}$$

This is simply the equation for an ellipse which is the cross-section of the beam deflected with a plane normal to the x -axis. Although this is anticipated intuitively, the important point is that the order of error is now specified. However, because the linear theory ignores the stretching of the centre-line, $(\partial h/\partial x)^2$ in equation (2.23) should be ignored. This is nothing but to regard the deflection as a translation of the beam in the y direction while neglecting the inclination, and the periphery of the cross-section normal to the x -axis remains to be circle. Hence use of the linear theory is justified with the proviso that terms of $\mathcal{O}(c^2/\lambda^2)$ are omitted. This order will be designated by $\mathcal{O}(\epsilon^2 \mu)$ later.

Using equation (2.23) with neglect of $(\partial h/\partial x)^2$, the kinematical condition (2.16) is given explicitly as follows:

$$(h - y) \left(\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial h}{\partial x} - \frac{\partial \phi}{\partial y} \right) + \frac{\partial \phi}{\partial z} z = 0 \quad \text{on } F = 0. \tag{2.24}$$

For this to be expressed in terms of the polar coordinates ($y = r \cos \theta$ and $z = r \sin \theta$), we only have to make the following substitutions:

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= \frac{\partial \phi}{\partial r} \cos \theta - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \sin \theta, \\ \frac{\partial \phi}{\partial z} &= \frac{\partial \phi}{\partial r} \sin \theta + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \cos \theta. \end{aligned} \tag{2.25}$$

3. ASYMPTOTIC EXPANSION

3.1. NORMALIZATION

At the outset, we start by normalizing the basic equations given in the preceding section. Using the typical magnitude of deflection and axial wavelength already introduced, all the dimensional variables on the left-hand sides below are replaced by the right-hand ones as follows:

$$\begin{aligned} (x, y, z, r, t) &\rightarrow (\lambda x, Ry, Rz, Rr, \lambda t/a_0), \\ (h, \phi, p - p_0, q) &\rightarrow \left(ch, \frac{a_0 c R}{\lambda} \phi, \frac{\rho_0 a_0^2 c R}{\lambda^2} p', \frac{\rho_0 a_0^2 c R^2}{\lambda^2} q \right). \end{aligned} \tag{3.1}$$

Because a typical time is measured by λ/a_0 , the normalization of ϕ is suggested by the fact that the radial velocity is caused by the temporal change of the deflection of the beam. The

normalization of the excess pressure $p - p_0$ is suggested by equation (2.7), where the pressure fluctuation is primarily brought about by $\rho_0 \partial \phi / \partial t$. Since we are concerned with weakly nonlinear and long waves propagating along the beam, the typical magnitude of deflection is much smaller than the tube radius R , while the typical wavelength λ is much longer than R . These assumptions are specified by the following two small parameters:

$$\epsilon = \frac{c}{R} \ll 1 \quad \text{and} \quad \mu = \left(\frac{R}{\lambda}\right)^2 \ll 1. \quad (3.2)$$

Note that assumption (2.20) is covered by the first assumption and α is regarded as being of the order of $\epsilon \sqrt{\mu}$. In the following, we discuss a case in which the values of these parameters are small but finite.

By normalizing equation (2.10), it is found that the linear acoustic wave equation is valid within the order of μ :

$$\mu \frac{\partial^2 \phi}{\partial t^2} - \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \phi - \mu \frac{\partial^2 \phi}{\partial x^2} = \mathcal{O}(\epsilon \mu). \quad (3.3)$$

Thus the leading nonlinearity in the flow field turns out to be of $\mathcal{O}(\epsilon \mu)$. Note that the terms with μ represent small contributions from air compressibility. The flexural wave equation (2.12) is made dimensionless as

$$\left(\frac{\partial}{\partial t} + M \frac{\partial}{\partial x} \right)^2 h + \mu J \frac{\partial^4 h}{\partial x^4} + Gh = \frac{\sigma}{\pi v^2} q \quad (3.4)$$

with q defined by

$$q = - \int_0^{2\pi} p'_b v \cos \theta' d\theta', \quad (3.5)$$

where p'_b denotes the normalized excess pressure on the beam surface, and the parameters M , J , G , σ and v are defined, respectively, as

$$M = \frac{U}{a_0}, \quad J = \frac{EI}{mR^2 a_0^2}, \quad G = \frac{K\lambda^2}{ma_0^2}, \quad \sigma = \frac{\pi \rho_0 b^2}{m}, \quad v = \frac{b}{R}. \quad (3.6)$$

Here, M is the Mach number of the travelling beam, and J , G and σ represent, respectively, the ratio of a typical speed of flexural waves $\sqrt{EI/mR^2}$ to the sound speed, the ratio of the natural angular frequency of the transverse motions of the beam due to the restoring force, $\omega_K (= \sqrt{K/m})$, to a frequency a_0/λ , and the ratio of a typical induced mass of the beam to the beam mass per unit axial length. Remember that the induced mass of the straight, circular cylinder placed in unbounded fluid of density ρ_0 is given by $\pi \rho_0 b^2$ per unit length for its transverse motion to the axis (Batchelor 1970). For reference, we evaluate these parameters in a plausible case. Suppose that a beam having $m = 2.4 \times 10^3$ kg/m, $EI = 2 \times 10^9$ kg m³/s², $K = 1.1 \times 10^5$ N/m² ($\omega_K/2\pi = 1.1$ Hz) and $b = 1.5$ m is travelling at $M = 0.4$ in a tube having $R = 5$ m. Under atmospheric pressure at room temperature, $a_0 = 340$ m/s and $\rho_0 = 1.2$ kg/m³, the parameters take the following numerical values: $J = 0.29$, $G = KR^2/ma_0^2\mu = 1.0 \times 10^{-2}/\mu$, $\sigma = 3.5 \times 10^{-3}$ and $v = 0.3$. Note that the definition of G differs from KR^2/ma_0^2 used in Sugimoto & Kugo (2001) by μ , so here G depends on λ and it becomes smaller for disturbances of shorter wavelength.

Next, we consider the boundary conditions. Condition (2.15) on the tube wall is imposed now at $r = 1$ as

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{at} \quad r = 1. \quad (3.7)$$

The other condition on the beam surface needs some remarks. Surface (2.23) is normalized on neglecting $(\partial h/\partial x)^2$ as follows:

$$(y - \epsilon h)^2 + z^2 - v^2 = \mathcal{O}(\epsilon^2 \mu). \tag{3.8}$$

We now want to express the periphery of the cross-section deflected in terms of the polar coordinates r and θ . According to the definition of θ' , we have

$$y = r \cos \theta = \epsilon h + v \cos \theta' + \mathcal{O}(\epsilon^2 \mu), \tag{3.9}$$

$$z = r \sin \theta = v \sin \theta' + \mathcal{O}(\epsilon^2 \mu). \tag{3.10}$$

Eliminating θ' between equations (3.9) and (3.10), r on the periphery of the cross-section is expressed in terms of θ as

$$r = v + \epsilon h \cos \theta - \frac{\epsilon^2 h^2}{2v} \sin^2 \theta + \mathcal{O}(\epsilon^3, \epsilon^2 \mu). \tag{3.11}$$

This is simply the result of the second cosine formula expanded by ϵ . Substituting this into equation (3.10), θ' is expressed in terms of θ as

$$\theta' = \theta + \frac{\epsilon h}{v} \sin \theta + \mathcal{O}(\epsilon^3, \epsilon^2 \mu), \tag{3.12}$$

invoking Taylor's expansion of $\sin \theta'$ around θ . Note that θ' does not include any terms of $\mathcal{O}(\epsilon^2)$. On this periphery, kinematical condition (2.24) is given, after normalization and expansion with respect to ϵ and μ , in the polar coordinates as

$$\frac{\partial \phi}{\partial r} - \frac{\partial h}{\partial t} \cos \theta + \epsilon \frac{h}{r} \left(\frac{\partial h}{\partial t} - \frac{\partial \phi}{\partial r} \cos \theta + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \sin \theta \right) = \mathcal{O}(\epsilon \mu). \tag{3.13}$$

3.2. ASYMPTOTIC EXPANSION

Since μ is small, we expand ϕ in terms of μ as follows:

$$\phi = \phi^{(0)} + \mu \phi^{(1)} + \mu^2 \phi^{(2)} + \dots, \tag{3.14}$$

where $\phi^{(n)}$ ($n = 0, 1, 2, \dots$) depend on r, θ, x and t . Because the problem is periodic with respect to θ and symmetric with respect to the plane $z = 0$, each $\phi^{(n)}$ may be expanded further into a Fourier cosine series as follows:

$$\phi^{(n)} = \sum_{m=0}^{\infty} \phi_m^{(n)} \cos m\theta, \tag{3.15}$$

with $\phi_m^{(n)} = \phi_m^{(n)}(r, x, t)$. In linear theory ($\epsilon \rightarrow 0$), only one harmonic with $m = 1$ suffices for the solution. The harmonics other than $m = 1$ are brought about by nonlinear effects. Note, therefore, that $\phi_m^{(n)}$ for $m \neq 1$ are necessarily accompanied by powers of ϵ . With expression (3.15) substituted into expression (3.14), the expansion results in a double asymptotic expansion with respect to μ and ϵ . Hence, it would be straightforward to expand ϕ , as ansatz, in the powers of ϵ and μ from the outset. But in order to carry the expansion up to higher order, many unnecessary expressions will then have to be written down. To avoid them, we expand ϕ with respect to μ first and then seek their coefficients in the expansion in terms of ϵ . In what follows, we determine the expansion of ϕ so that the lowest contributions of $\mathcal{O}(\epsilon^2, \mu)$ may be included in the wave equation desired.

Substituting expression (3.14) into equation (3.3), we have from the terms of $\mathcal{O}(1)$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\right) \phi^{(0)} = 0. \tag{3.16}$$

Decomposing $\phi^{(0)}$ into the Fourier series, it follows from (3.16) that

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{m^2}{r^2}\right) \phi_m^{(0)} = 0. \tag{3.17}$$

The solutions $\phi_m^{(0)}$ are obtained immediately as

$$\phi_m^{(0)} = \begin{cases} f_0^{(0)} + g_0^{(0)} \ln r, & (m = 0), \\ r^m f_m^{(0)} + \frac{g_m^{(0)}}{r^m}, & (m = 1, 2, 3, \dots), \end{cases} \tag{3.18}$$

where $f_m^{(0)}(x, t)$ and $g_m^{(0)}(x, t)$ ($m = 0, 1, 2, \dots$) are unknown functions of x and t .

For each Fourier component, the boundary condition on the tube wall requires that

$$\frac{\partial \phi_m^{(0)}}{\partial r} = 0 \quad \text{at } r = 1, \tag{3.19}$$

where $m = 0, 1, 2, \dots$. Imposing these boundary conditions, $g_m^{(0)}$ are expressed in terms of $f_m^{(0)}$, and $\phi_m^{(0)}$ are given as follows:

$$\phi_m^{(0)} = \begin{cases} f_0^{(0)}(x, t), & (m = 0), \\ \left(r^m + \frac{1}{r^m}\right) f_m^{(0)}(x, t), & (m = 1, 2, 3, \dots). \end{cases} \tag{3.20}$$

Boundary condition (3.13) on beam surface (3.11) is given, upon neglect of $\mathcal{O}(\epsilon\mu)$, as

$$\frac{\partial \phi^{(0)}}{\partial r} - \frac{\partial h}{\partial t} \cos \theta = -\epsilon \frac{h}{r} \left(\frac{\partial h}{\partial t} - \frac{\partial \phi^{(0)}}{\partial r} \cos \theta + \frac{1}{r} \frac{\partial \phi^{(0)}}{\partial \theta} \sin \theta \right). \tag{3.21}$$

Since ϵ is small, this condition suggests that $\phi^{(0)}$ should be sought in the following form:

$$\phi^{(0)} = \phi_1^{(0)} \cos \theta + \epsilon(\phi_0^{(0)} + \phi_2^{(0)} \cos 2\theta) + \epsilon^2 \phi_3^{(0)} \cos 3\theta + \mathcal{O}(\epsilon^3). \tag{3.22}$$

But we ignore the terms of $\mathcal{O}(\epsilon)$, for a moment, to look for the lowest-order relation. Then equation (3.21) is reduced to

$$\frac{\partial \phi_1^{(0)}}{\partial r} = \frac{\partial h}{\partial t} + \mathcal{O}(\epsilon^2) \quad \text{at } r = v. \tag{3.23}$$

Substituting $\phi_1^{(0)}$ into this, $f_1^{(0)}$ is expressed in terms of h as

$$f_1^{(0)} = -\frac{v^2}{1-v^2} \frac{\partial h}{\partial t} + \mathcal{O}(\epsilon^2). \tag{3.24}$$

This is simply the linear solution.

We now proceed to take ϕ up to the next higher-order terms of $\mathcal{O}(\epsilon, \mu)$ as

$$\begin{aligned} \phi = & \left(r + \frac{1}{r}\right) f_1^{(0)} \cos \theta + \epsilon \left[f_0^{(0)} + \left(r^2 + \frac{1}{r^2}\right) f_2^{(0)} \cos 2\theta \right] \\ & + \mu \phi_1^{(1)} \cos \theta + \mathcal{O}(\epsilon^2, \epsilon\mu, \mu^2). \end{aligned} \tag{3.25}$$

Substituting this into equation (3.3), we have from the coefficient of $\cos \theta$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}\right) \phi_1^{(1)} = \left(r + \frac{1}{r}\right) \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) f_1^{(0)}. \tag{3.26}$$

Using the boundary condition on the tube wall, we obtain

$$\phi_1^{(1)} = \left(\frac{r^3}{8} + \frac{r}{2} \ln r - \frac{7}{8} r\right) \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) f_1^{(0)} + \left(r + \frac{1}{r}\right) f_1^{(1)}, \tag{3.27}$$

where $f_1^{(1)}(x, t)$ is arbitrary.

Next we impose the boundary condition on the beam surface. After substitution of expression (3.25) into condition (3.13), and setting r equal to that given by expression (3.11), we expand the condition into the Fourier series. After a little lengthy calculation, we have from the coefficient of $\cos \theta$

$$\begin{aligned} &\left(1 - \frac{1}{v^2}\right) (f_1^{(0)} + \mu f_1^{(1)}) - \frac{\partial h}{\partial t} + \epsilon^2 \left[\left(\frac{1}{v^2} - \frac{4}{v^4}\right) h^2 f_1^{(0)} - \left(1 - \frac{3}{v^4}\right) h f_2^{(0)} - \frac{h^2}{v^2} \frac{\partial h}{\partial t} \right] \\ &+ \mu \left(\frac{3v^2}{8} + \frac{1}{2} \ln v - \frac{3}{8}\right) \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) f_1^{(0)} = \mathcal{O}(\epsilon^4, \epsilon^2 \mu). \end{aligned} \tag{3.28}$$

From the coefficient of $\cos 2\theta$, we obtain

$$\frac{h}{v^4} f_1^{(0)} + \left(1 - \frac{1}{v^4}\right) f_2^{(0)} = \mathcal{O}(\epsilon^2, \mu). \tag{3.29}$$

Solving $f_1^{(0)} + \mu f_1^{(1)}$ and $f_2^{(0)}$ from relations (3.28) and (3.29), and using the lowest relation (3.24), we have

$$\begin{aligned} f_1^{(0)} + \mu f_1^{(1)} = & -\frac{v^2}{1-v^2} \frac{\partial h}{\partial t} - \frac{\mu v^4}{(1-v^2)^2} \left(\frac{1}{2} \ln v - \frac{3}{8} + \frac{3v^2}{8}\right) \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) \frac{\partial h}{\partial t} \\ & - \frac{2\epsilon^2 v^4}{(1-v^2)^3(1+v^2)} h^2 \frac{\partial h}{\partial t} + \mathcal{O}(\epsilon^4, \epsilon^2 \mu) \end{aligned} \tag{3.30}$$

and

$$f_2^{(0)} = -\frac{v^2}{(1-v^2)^2(1+v^2)} h \frac{\partial h}{\partial t} + \mathcal{O}(\epsilon^2). \tag{3.31}$$

Thus the higher-order terms neglected in relation (3.24) are now specified simultaneously in the process of determining the lowest expression of $f_2^{(0)}$. For the component independent of θ , incidentally, the boundary condition is automatically satisfied within the order of ϵ by use of relation (3.24) so that $f_0^{(0)}$ is left undetermined within the present approximation. It will turn out shortly that the contribution from $f_0^{(0)}$ to the total force acting on the beam vanishes. Therefore, we do not pursue $f_0^{(0)}$ by going into higher-order relations.

Following the same step-by-step method, we will be able to carry expansion (3.14) up to any order desired. As we proceed, unknown higher-order coefficients are determined, while higher-order corrections to the coefficients so far obtained are determined consistently.

3.3. EVALUATION OF THE PRESSURE FORCE and DERIVATION OF WAVE EQUATION

With ϕ available, we proceed to evaluate the pressure force q acting on the beam. Before doing so, equation (2.7) must be made dimensionless according to the replacement of equation (3.1). Expanding equation (2.7) thus normalized in terms of ϵ and μ , the

dimensionless excess pressure p' is given by

$$p' = -\frac{\partial\phi}{\partial t} - \frac{\epsilon}{2} \left[\left(\frac{\partial\phi}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial\phi}{\partial\theta} \right)^2 \right] + \mathcal{O}(\epsilon\mu). \tag{3.32}$$

The excess pressure on the beam surface p'_b is evaluated as a function of θ by substitution of relation (3.11) for r in equation (3.32) and is given by

$$\begin{aligned} p'_b = & \left[\frac{v(1+v^2)}{1-v^2} \frac{\partial^2 h}{\partial t^2} - \frac{\epsilon^2(1-v^2-17v^4+v^6)}{8v(1-v^2)^3} h^2 \frac{\partial^2 h}{\partial t^2} - \frac{\epsilon^2(1-v^2-2v^4)}{v(1-v^2)^3} h \left(\frac{\partial h}{\partial t} \right)^2 \right. \\ & \left. - \frac{\mu v^3}{(1-v^2)^2} \left(\ln \frac{1}{v} + \frac{5}{4} - v^2 - \frac{v^4}{4} \right) \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \frac{\partial^2 h}{\partial t^2} \right] \cos\theta \\ & - \epsilon \left[\frac{\partial f_0^{(0)}}{\partial t} + \frac{h}{2} \frac{\partial^2 h}{\partial t^2} + \frac{1+v^4}{2(1-v^2)^2} \left(\frac{\partial h}{\partial t} \right)^2 \right] + \epsilon \left[\frac{1+v^2+3v^4-v^6}{2(1-v^2)^2(1+v^2)} h \frac{\partial^2 h}{\partial t^2} \right. \\ & \left. + \frac{1+v^2+2v^4}{(1-v^2)^2(1+v^2)} \left(\frac{\partial h}{\partial t} \right)^2 \right] \cos 2\theta + \mathcal{O}(\epsilon^2) \cos 3\theta. \end{aligned} \tag{3.33}$$

The total force q is obtained by integrating p'_b along the periphery of the cross-section with respect to θ' as

$$q = - \int_0^{2\pi} v p'_b \cos\theta' d\theta'. \tag{3.34}$$

Since p'_b is now available as the function of θ , relation (3.12) between θ' and θ is used to change the variable θ' to θ as

$$\cos\theta' d\theta' = \left[\left(1 - \frac{3\epsilon^2 h^2}{8v^2} \right) \cos\theta + \frac{\epsilon h}{v} \cos 2\theta \right] d\theta + \mathcal{O}(\epsilon^2) \cos 3\theta. \tag{3.35}$$

With this substitution, the integral is executed with respect to θ from 0 to 2π . Then q is obtained as

$$\begin{aligned} \frac{q}{\pi v^2} = & - \left(\frac{1+v^2}{1-v^2} \right) \frac{\partial^2 h}{\partial t^2} - \frac{2\epsilon^2 v^2}{(1-v^2)^3(1+v^2)} h \frac{\partial^2 h^2}{\partial t^2} \\ & + \frac{\mu v^2}{(1-v^2)^2} \left[\ln \frac{1}{v} + \frac{(1-v^2)(5+v^2)}{4} \right] \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \frac{\partial^2 h}{\partial t^2} + \mathcal{O}(\epsilon^4, \epsilon^2\mu, \mu^2). \end{aligned} \tag{3.36}$$

By substitution of this into equation (3.4), we derive the following nonlinear wave equation for h :

$$\left(\frac{\partial}{\partial t} + M \frac{\partial}{\partial x} \right)^2 h + \mu J \frac{\partial^4 h}{\partial x^4} + Gh = -\sigma s \frac{\partial^2 h}{\partial t^2} - \epsilon^2 \sigma \alpha h \frac{\partial^2 h^2}{\partial t^2} + \mu \sigma \eta \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \frac{\partial^2 h}{\partial t^2}, \tag{3.37}$$

where s , α and β are defined as

$$s = \frac{1+v^2}{1-v^2}, \quad \alpha = \frac{2v^2}{(1-v^2)^3(1+v^2)} \quad \text{and} \quad \eta = \frac{v^2}{(1-v^2)^2} \left[\ln \frac{1}{v} + \frac{(1-v^2)(5+v^2)}{4} \right]. \tag{3.38}$$

This is the desired equation to describe the propagation of flexural waves on the elastic beam under the nonlinear aerodynamic loading.

4. DISCUSSIONS

The analysis has been developed so that the first-order corrections of ϵ and μ may be included in the resultant wave equation (3.37). Because of the cubic nonlinearity, the correction is of order ϵ^2 not by ϵ . It should be remarked that the corrections are greater than the errors of $\mathcal{O}(\epsilon^2\mu)$ inherent in the use of the linear flexural wave equation.

It is revealed from equation (3.37), and also relation (3.36), that the aerodynamic loading consists of three terms on the right-hand side, all of which are accompanied by the parameter σ . As expected, the coefficient σs of the first term is the induced mass (relative to m) of the straight beam confined in the concentric tube (Païdoussis 1998). The factor s reflects the effect of the tube wall. In fact, as the beam becomes thin, i.e., $v \rightarrow 0$, s tends to unity and σs approaches the induced mass in unbounded fluid. The nonlinearity due to finite deflection arises directly from the kinematical condition on the beam surface; but it may be attributed to the presence of the tube wall at finite distance. In fact, the coefficient α disappears in the limit as $v \rightarrow 0$.

When the limit $\mu \rightarrow 0$ is taken in relation (3.36), the result gives the fluid dynamic loading on a straight rigid beam executing uniform, transverse motions in an incompressible fluid [of course, it is also valid for a beam at rest in the axial direction ($M = 0$)]. This force is also expressed in dimensional form as follows:

$$-\pi\rho_0 b^2 \left(\frac{1+v^2}{1-v^2} \right) \frac{\partial^2 h}{\partial t^2} - \frac{2\pi\rho_0 v^4}{(1-v^2)^3(1+v^2)} h \frac{\partial^2 h^2}{\partial t^2}. \tag{4.1}$$

Developing the derivative of h^2 , the term with $\partial^2 h/\partial t^2$ contributes substantially to an increase of the induced mass by $2\pi\rho_0 b^2 \alpha (h/R)^2$, while the term with $h(\partial h/\partial t)^2$ introduces the force proportional to the product of h and the square of the velocity, which changes its direction depending on the sign of deflection. Note that the second term in relation (4.1) becomes pronounced as $v \rightarrow 1$. In passing, it is verified that the rate of change of the total kinetic energy of the fluid in the annular region per unit axial length, denoted by E , balances with the power input by the total force exerted on the fluid $-q$ times the velocity of the beam $\partial h/\partial t$:

$$\frac{dE}{dt} = -q \frac{\partial h}{\partial t}, \tag{4.2}$$

where E is defined by

$$\begin{aligned} E &= \frac{1}{2} \int_0^{2\pi} d\theta \int_{r_b}^1 \left[\left(\frac{\partial \phi}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \phi}{\partial \theta} \right)^2 \right] r dr \\ &= \frac{\pi v^2}{2} \left[\frac{1+v^2}{1-v^2} + \frac{4\epsilon^2 v^2 h^2}{(1-v^2)^3(1+v^2)} \right] \left(\frac{\partial h}{\partial t} \right)^2, \end{aligned} \tag{4.3}$$

where r_b denotes r given by relation (3.11). Note that E is not only proportional to the velocity squared but is dependent on h^2 . It is also verified that the rate of change of the total momentum of fluid in the y direction, denoted by M , balances with the pressure forces not only due to the beam but also to the tube wall. In fact, we have

$$\frac{dM}{dt} = q_{\text{wall}} - q, \tag{4.4}$$

where

$$M = \int_0^{2\pi} d\theta \int_{r_b}^1 \left(\frac{\partial \phi}{\partial r} \cos \theta - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \sin \theta \right) r dr = -\pi v^2 \frac{\partial h}{\partial t}, \tag{4.5}$$

and q_{wall} is given by

$$q_{\text{wall}} = - \int_0^{2\pi} p' \cos \theta \, d\theta = -2\pi \left[\frac{v^2}{1-v^2} \frac{\partial^2 h}{\partial t^2} + \frac{\epsilon^2 v^4}{(1-v^2)^3(1+v^2)} h \frac{\partial^2 h^2}{\partial t^2} \right], \tag{4.6}$$

at $r = 1$.

The effects of air compressibility and of axial curvature of the beam are taken into account through the last term of equation (3.37). If both σ and G are small enough, the first term on the left-hand side remains the leading term. This implies that h is given by $h(x - Mt)$ to the lowest order; i.e., the deflection is stationary with the beam. Using this, i.e., replacing $\partial h/\partial t$ with $-M\partial h/\partial x$, the last term on the right-hand side may be approximated as

$$\mu\sigma\eta \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \frac{\partial^2 h}{\partial t^2} \approx -\mu\sigma\eta(1 - M^2)M^2 \frac{\partial^4 h}{\partial x^4}. \tag{4.7}$$

This term may now be incorporated into the bending term on the left-hand side of equation (3.37) as

$$\left(\frac{\partial}{\partial t} + M \frac{\partial}{\partial x} \right)^2 h + \mu J_\sigma \frac{\partial^4 h}{\partial x^4} + Gh = -\sigma s \frac{\partial^2 h}{\partial t^2} - \epsilon^2 \sigma \alpha h \frac{\partial^2 h^2}{\partial t^2}, \tag{4.8}$$

with J_σ defined by

$$J_\sigma = J + \sigma\eta(1 - M^2)M^2. \tag{4.9}$$

It is found that for $M < 1$, the effects of compressibility and of axial curvature act to increase the bending rigidity.

Let us verify the validity of the linear terms of equation (3.37) by comparing the dispersion relation with that of the exact theory. The full linear dispersion relation was derived previously without making the assumption $\mu \ll 1$ (Sugimoto 1996). Assuming an elementary solution in the form proportional to $\exp[i(kx - \omega t)]$, k and ω being a dimensionless wavenumber and a dimensionless frequency normalized, respectively, by R^{-1} and a_0/R , it is given as follows:

$$[(\omega - Mk)^2 - Jk^4 - \mu G]\beta^2 D_4 = (\sigma/v)\beta D_3 \omega^2, \tag{4.10}$$

with $\beta^2 = k^2 - \omega^2$ and D_3 and D_4 defined by

$$\begin{aligned} D_3 &= I_1(\beta v)K'_1(\beta) - K_1(\beta v)I'_1(\beta), \\ D_4 &= I'_1(\beta v)K'_1(\beta) - K'_1(\beta v)I'_1(\beta), \end{aligned} \tag{4.11}$$

where I_1 and K_1 denote the modified Bessel functions of first order and the prime denotes differentiation with respect to the argument. Figure 3 shows graphically the dispersion curves given by (4.10) for $M = 0.4$, $J = 0.29$, $\mu G = 0.01$, $\sigma = 3.5 \times 10^{-3}$ and $v = 0.3$, where the dotted line represents $\omega = Mk$. For a real $k \geq 0$, there are many, real solutions of ω to equation (4.10), among which the four branches in $|\omega| \leq 5$ are drawn for $0 \leq k \leq 5$, and the curves for $k < 0$ are symmetric with respect to the origin. Although the curves appear to cross each other, they are bent sharply without crossing.

Since we are concerned with the long waves and $\mu G \ll 1$, we expand equation (4.10) around $k = 0$. Then $D_3/(\beta D_4)$ is expanded for $|\beta| \ll 1$ as [see, e.g., Abramowitz & Stegun (1972)]

$$\frac{D_3}{\beta D_4} = -v \left(\frac{1+v^2}{1-v^2} \right) + \frac{v^3}{(1-v^2)^2} \left[\ln \frac{1}{v} + \frac{(1-v^2)(5+v^2)}{4} \right] \beta^2 + \mathcal{O}(\beta^4). \tag{4.12}$$

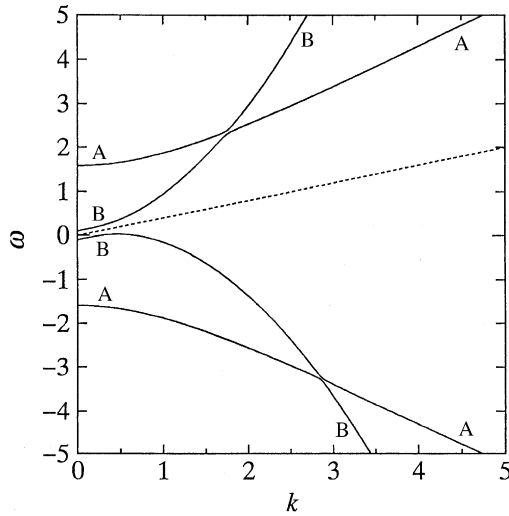


Fig. 3. Graph of the full dispersion curves given by equation (4.10) for $M = 0.4$, $J = 0.29$, $\mu G = 0.01$, $\sigma = 3.5 \times 10^{-3}$ and $\nu = 0.3$ where the dotted line represents the line $\omega = Mk$ and the labels 'A' and 'B' indicate the acoustic and beam modes, respectively, in the case of $\sigma = 0$.

Using this, the dispersion relation (4.10) is expanded as

$$(\omega - Mk)^2 - \mu Jk^4 - G = -\sigma s\omega^2 - \mu\sigma\eta(\omega^2 - k^2)\omega^2, \tag{4.13}$$

where k and ω in equation (4.10) have been replaced by $\mu^{1/2}k$ and $\mu^{1/2}\omega$, respectively, to be consistent with normalization (3.1). This dispersion relation agrees with that of equation (3.37) for a monochromatic wave of h in the form of $\exp[i(kx - \omega t)]$. This also endorses the validity of the expansion.

5. CONCLUSION

The nonlinear aerodynamic loading on the lateral surface of the elastic beam travelling in the air-filled tube has been examined, and the total pressure force on the beam q is obtained as (3.36). As far as flexural motions of long wavelength are concerned, the flow field may be treated by linear acoustic theory, and the only source of nonlinearity lies in the kinematical condition on the beam surface. But as the coefficient of the nonlinear term tends to vanish for a slender beam, the nonlinearity may be attributed to the presence of the tube wall. It is found that the nonlinearity acts not only to increase the induced mass but also to introduce the additional force proportional to the product of the deflection and the square of the lateral velocity of the beam. The effect of axial curvature of the beam due to long but finite wavelength has also been examined. It is found that this effect appears in the form of the fourth-order derivative of the deflection, in which the lateral acceleration of the beam is multiplied by the acoustic wave operator in the axial direction.

Using the aerodynamic loading thus obtained, and ignoring the end-effects, the nonlinear wave equation (3.37) and its simplified version (4.8) in the case of weak restoring force ($G \ll 1$) have been derived. Although the effects included are small in the equation, it should be remarked that they will accumulate to manifest themselves significantly in the long-time behaviour of the system.

REFERENCES

- ABRAMOWITZ, M. & STEGUN, I. A. 1972 *Handbook of Mathematical Functions*. New York: Dover Publications.
- BATCHELOR, G. K. 1970 *An Introduction to Fluid Dynamics*. Cambridge: Cambridge University Press.
- HOWE, M. S. 1986 Attenuation and diffraction of bending waves at gaps in fluid loaded plates. *IMA Journal of Applied Mathematics* **36**, 247–262.
- KITTEL, C. 1976 *Introduction to Solid State Physics*. New York: Wiley.
- PAÏDOUSSIS, M. P. 1998 *Fluid-Structure Interactions, Slender Structures and Axial Flow*, Vol.1. San Diego: Academic Press.
- SUGIMOTO, N. 1996 Aeroelastic stability of a beam traveling in a tunnel lined with resonators. *AIAA Journal* **34**, 2005–2013.
- SUGIMOTO, N. & HORIOKA, T. 1995 Dispersion characteristics of sound waves in a tunnel with an array of Helmholtz resonators. *Journal of the Acoustical Society of America* **97**, 1446–1459.
- SUGIMOTO, N. & KUGO, K. 2001 Wave propagation on an elastic beam traveling in a tube: linear theory of aerodynamic loading. *Journal of Computational Acoustics* **9**, 1227–1236.